# Tracking Points While Rolling Without Slipping

Anthony Wang

12 January 2024

## §1 Rolling Without Slipping

**Definition 1.1.** Given an object[1] $R$ and a fixed curve $\gamma$ initially tangent to it, we say that $R$ *rolls without slipping* on $\gamma$ if $R$ moves[2] such that some point on the boundary of $R$ is always tangent to $\gamma$ and **when any point on $R$ is this tangency point, it is stationary.**

Notably, we make no distinction to any "point" that the object rotates about.

## §2 Tracking a Marked Point

Let's see some implications of this definition to tracking a marked point on an object that is rolling without slipping.

**Proposition 2.1**

Mark a point $P$ on the object, and suppose that it moves on a path $\Gamma_P$. Furthermore, suppose that the object is tangent to $\gamma$ at $T$. Then, $PT$ is normal to $\Gamma_P$.

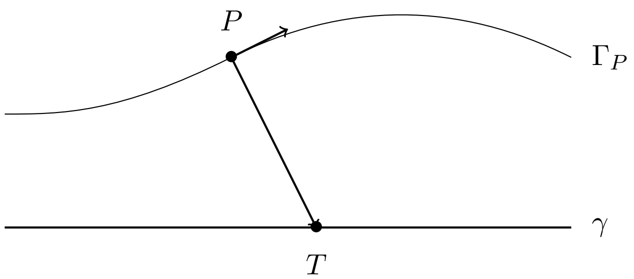


Figure 1: Marked point $P$ moving along path $\Gamma_P$

*Proof.* Let $P(t)$ denote the position of $P$ at time $t$. Let $T(t)$ denote the position of $T$ at time $t$. We consider both $P$ and $T$ to move with the object.

[1]We will represent objects as the interiors of smooth, simple, closed, planar curves

[2]i.e. moves rigidly preserving orientation. You can think about this as a path through the space of rototranslations starting at the identity.

Suppose the object is tangent to $\gamma$ at $T$ at the time $t = t_0$. By definition, we have $T'(t_0) = 0$. Now note that, $P'(t_0)$ is a tangent vector to $\Gamma_P$ at $P$ (at time $t = t_0$). In addition,

$$P'(t_0) \cdot (T(t_0) - P(t_0)) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(T(t) - P(t)) \cdot (T(t) - P(t))\right]\Big|_{t=t_0}$$
$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|T(t) - P(t)\|^2\Big|_{t=t_0}.$$

But since the object moves rigidly, $\|T(t)-P(t)\|$ is a constant, so $P'(t_0)\cdot(T(t_0)-P(t_0)) = 0$, as desired.

□

Furthermore, since $\|T(t) - P(t)\|$ is does not depend on $t$, and $T(t)$ is momentarily stationary, we should be able to model the movement of $P$ when the time is near $t$ as following the circle centered at $T(t)$ passing through $P(t)$. Unfortunately, this is not exactly true (e.g. though they are tangent, the curvatures of said circle and $\Gamma_P$ are not always equal at $P$).

However, if we let $s(t)$ denote the arc length traversed by $P$ on $\Gamma_P$ from time 0 to time $t$, and let $\theta(t)$ denote the angle rotated by the object from time 0 to time $t$ (e.g. by marking a fixed line on the object, then tracking the angle the $x$-axis makes with it), we have

**Theorem 2.2** (Arc Length of Marked Points)

Suppose $r(t)$ is the distance from $P(t)$ to the tangency point of the object with $\gamma$. Then

$$r = \frac{\mathrm{d}s}{\mathrm{d}\theta},$$

considering everything as a function of $t$.

In other words, this states that if the object rotates by an infinitesimal angle of $\mathrm{d}\theta$, then a point $P$ that is $r$ away from the tangency point of the object with $\gamma$ will move by $r\,\mathrm{d}\theta$.

*Proof of the Theorem.* This proof is a bit computational. We will show that, using notation from 2.1, at time $t = t_0$

$$\|T(t_0) - P(t_0)\| = \frac{\mathrm{d}s}{\mathrm{d}\theta}\Big|_{t=t_0}.$$

Let $r$ be the **constant** equal to $\|T(t_0) - P(t_0)\|$, and let $\hat{\imath}$ be the unit vector in the positive $x$ direction. Finally, define

$$\hat{n}(t) := \frac{1}{r}(T(t) - P(t)).$$

It's easy to see that $\cos^{-1}(\hat{n}(t) \cdot \hat{\imath})$ is the angle that the line $PT$ makes with the $x$-axis at time $t$, so it is off from $\theta(t)$ by a constant[3]. Thus,

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\cos^{-1}(\hat{n} \cdot \hat{\imath})}{\mathrm{d}s} = -\frac{1}{\sqrt{1 - (\hat{n} \cdot \hat{\imath})^2}} \cdot \frac{\mathrm{d}(\hat{n} \cdot \hat{\imath})}{\mathrm{d}s}.$$

[3]Or off from $-\theta(t)$ by a constant. It's iffy because we ideally want $\theta$ to be directed, but our dot product can only detect undirected angles. We can simply pretend all rotations are $\ll \frac{1}{2}\tau$, and that $\theta(t)$ is always defined so that $\frac{\mathrm{d}s}{\mathrm{d}\theta}$ is positive.

We can check that

$$\frac{\mathrm{d}(\hat{n}\cdot\hat{\imath})}{\mathrm{d}s} = (\hat{n}'(t)\cdot\cdot\hat{\imath})\frac{\mathrm{d}t}{\mathrm{d}s} = -\left(\frac{P'(t)}{r}\cdot\hat{\imath}\right)\frac{\mathrm{d}t}{\mathrm{d}s} = -\frac{1}{r}\left(\frac{\mathrm{d}P}{\mathrm{d}s}\cdot\hat{\imath}\right).$$

But note that $\frac{\mathrm{d}P}{\mathrm{d}s}\Big|_{t=t_0}$ is the unit tangent vector to $\gamma_P$ at $P(t_0)$, so by Proposition 2.1, it and $\hat{n}$ form an orthonormal basis of $\mathbb{R}^2$ [4]. It follows that

$$(\hat{n}(t_0)\cdot\hat{\imath})^2 + \left(\frac{\mathrm{d}P}{\mathrm{d}s}\bigg|_{t=t_0}\cdot\hat{\imath}\right)^2 = \|\hat{\imath}\|^2 = 1.$$

Therefore,

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}\bigg|_{t=t_0} = \frac{1}{r(t_0)},$$

as desired. □

## §3 Example Problems

**Problem 3.1** (Physics)**.** Suppose a circle with radius $r$ rolls without slipping on a line such that its center moves with velocity $v$ and the entire circle rotates about its center with an angular velocity of $\omega$. Then $v = r\omega$.

*Proof.* Track the position of the center. Note that it moves on a line at constant velocity and is always a distance $r$ from the tangency point. Therefore, by Theorem 2.2,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{r\mathrm{d}\theta}{\mathrm{d}t} = r\omega.$$

□

**Problem 3.2** (Folklore)**.** Roll a coin around the circumference of another (fixed) coin of equal radius. How many times does the moving coin rotate?

*Proof.* Suppose the shared radius if $r$. Again, track the position of the center of the rolling object. Note that it moves on a circle with radius $2r$ concentric with the fixed coin. Therefore, by Theorem 2.2,

$$\Delta\theta = \int \mathrm{d}\theta = \int \frac{1}{r}\,\mathrm{d}s = \frac{1}{r}\Delta S = \frac{1}{r}\cdot\tau\cdot 2r = 2\tau,$$

so the coin has rotated twice. □

**Problem 3.3** (Delaunay)**.** An ellipse with semimajor axis $a$ rolls without slipping along the $x$-axis for one complete turn. Find the length of curve traced out by one focus $F$.

Surprisingly (or unsurprisingly), this is independent of the eccentricity!

*Proof.* By Theorem 2.2,

$$S = \int \mathrm{d}s = \int_0^{\tau} r\,\mathrm{d}\theta.$$

But by symmetry, for any $\phi \in [0, \frac{1}{2}\tau)$, the marked point on the ellipse tangent to the $x$-axis when $\theta = \phi$ (i.e. $T(t_1)$ for $\theta(t_1) = \phi$) is the reflection over the center of the ellipse of the marked point on the ellipse tangent to the $x$-axis when $\theta = \phi + \frac{1}{2}\tau$. In other words,

[4] In fact, these are the tangent and normal unit vectors of the Frenet—Serret frame!

the moments in which the ellipse is tangent to the $x$-axis at diametrically opposite points are a half turn away.

But the sum of the distances from two diametrically opposite points to a fixed focus $F$ is simply $2a$ since the two foci and the two diametrically opposite points form a parallelogram. Therefore,

$$2S = \int_0^\tau 2a \, \mathrm{d}\theta = 2a\tau.$$

It follows that curve has length $a\tau$. □